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Irreducible Ideals of Finitely Generated Commutative Monoids[1]

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We give a characterization of irreducible ideals of finitely generated commutative monoids and show how to find a decomposition of an ideal into irreducibles.

INTRODUCTION

All monoids and rings appearing in this work are commutative; for this reason in the rest of the paper we omit this adjective. $\mathbb{N}$ denotes the set of nonnegative integers.

Decomposition of an ideal into irreducibles has been mainly used in the literature for proving the existence of primary decompositions of ideals of rings and monoids (see for instance [1, 2]). There are procedures for computing primary decompositions of ideals in affine algebras and methods for deciding whether a given ideal is primary (see [3]). Nevertheless it seemed that the study of irreducible ideals and decompositions into them had been almost forgotten until [4]. In this paper we focus our attention into the characterization of irreducible ideals of finitely generated monoids and in giving a procedure for finding a decomposition of an ideal into irreducibles.

Though there exists a parallelism between the ideal theory in rings and in monoids (see for instance [1]), the slight differences between these two theories are the ones that make this work not suitable (at least at first sight) for the study of irreducibles in rings (the main divergence lies in Lemma 9).

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The contents of this work are organized as follows. First we concentrate on the study of irreducible ideals and decomposition into irreducibles of ideals of $\mathbb{N}^p$. After this we generalize these results for arbitrary finitely generated monoids and finally we discuss some computational aspects for finding such decompositions.

1. PRELIMINARIES

Let S be the monoid generated by $\{s_1, \ldots, s_p\}$. Define

$$\varphi\colon \mathbb{N}^p \to S, \qquad \varphi(a_1, \ldots, a_p) = \sum_{i=1}^{p} a_i s_i$$

and denote by σ the kernel congruence of φ. Then S is isomorphic to $\mathbb{N}^p/\sigma$. The σ-class of an element $x \in \mathbb{N}^p$ will be denoted by $[x]_\sigma$, and if there is no possible misunderstanding, by $[x]$.

Given a subset A of S, denote the set $\varphi^{-1}(A)$ by $E(A)$, that is,

$$E(A) = \{x \in \mathbb{N}^p \mid \varphi(x) \in A\}$$

(the set of expressions of elements in A).

A set ρ is a *system of generators* of σ (as a congruence) if σ is the least congruence containing ρ. In this setting we say that ρ is a *presentation* of S.

From ρ one can construct by using critical pair completion a canonical system of generators $\kappa = \{(\alpha_1, \beta_1), \ldots, (\alpha_l, \beta_l)\}$ of σ with respect to a given linear admissible order $\preceq$ on $\mathbb{N}^p$ (see [6] for a more general approach [8]). This new system of generators of σ allows us to construct the map $\mathrm{NF}_\kappa\colon \mathbb{N}^p \to \mathbb{N}^p$ as follows:

(1) if $x - \alpha_i \notin \mathbb{N}^p$ for all $i \in \{1, \ldots, l\}$, then $\mathrm{NF}_\kappa(x) = x$;

(2) if $x - \alpha_j \notin \mathbb{N}^p$ for all $j \leq i$ and $x - \alpha_{i+1} \in \mathbb{N}^p$, then $\mathrm{NF}_\kappa(x) = \mathrm{NF}_\kappa(x - \alpha_{i+1} + \beta_{i+1})$.

One can prove that $\mathrm{NF}_\kappa(x) = \min_\preceq [x]_\sigma$ ($[x]_\sigma$ denotes the σ-class of x in $\mathbb{N}^p$; if there is no possible misunderstanding, we will simply write $[x]$), and therefore $x \sigma y$ if and only if $\mathrm{NF}_\kappa(x) = \mathrm{NF}_\kappa(y)$. In this way canonical systems of generators are a tool for solving the word problem on finitely generated monoids.

A subset I of S is an *ideal* if $I + S \subseteq I$ (that is, for all $x \in I$ and $s \in S$, the element $x + s$ is in I). An ideal I is *generated* by $A \subseteq S$ if $I = A + S$. We say that the ideal I is *finitely generated* if there exists $A \subseteq S$ finite such that $I = A + S$, and it is *principal* if there exists $s \in S$ such that

$I = \{s\} + S$; in this case we usually write $s + S$ instead of $\{s\} + S$. An ideal I of S is *irreducible* if there exist no ideals J, K of S properly containing I such that $I = J \cap K$.

Associated with an ideal I of S we can define the *Rees congruence* $\mathscr{R}_I$ as $a\mathscr{R}_I b$ if either $\{a, b\} \subset I$ or $a = b$. In the monoid quotient $S/\mathscr{R}_I$ all the elements in I are identified. Once we know a presentation ρ of S and a set $B = \{\lambda_1, \dots, \lambda_r\}$ such that $\{\varphi(\lambda_1), \dots, \varphi(\lambda_r)\}$ generates I, we can compute a presentation of $S/\Re_I$ in the following way. Set

$$\rho_I = \{(\lambda_1, \lambda_2), \dots, (\lambda_1, \lambda_r), (\lambda_1 + e_1, \lambda_1), \dots, (\lambda_1 + e_p, \lambda_1)\} \cup \rho,$$

and let σ_I be the congruence on $\mathbb{N}^p$ generated by ρ_I. Then $S/\mathscr{R}_I$ is isomorphic to $\mathbb{N}^p/\sigma_I$ (the proof of this fact is not hard and can be found in [7]).

Given $a, b \in S$, we write $a \leq b$ whenever $a + c = b$ for some $c \in S$.

2. IRREDUCIBLE IDEALS OF $\mathbb{N}^p$

This section is devoted to the study of irreducible ideals of $\mathbb{N}^p$. We start by giving a characterization of the irreducible ideals of $\mathbb{N}^p$ and then we show how to compute a decomposition into irreducibles of a given ideal of $\mathbb{N}^p$.

Since intersections of ideals are involved in our computations, we need to learn how to calculate the intersection of two ideals in $\mathbb{N}^p$ once we are given its systems of generators. For $a = (a_1, \dots, a_p)$, $b = (b_1, \dots, b_p) \in \mathbb{N}^p$, set

$$a \vee b = (\max\{a_1, b_1\}, \dots, \max\{a_p, b_p\})$$

and

$$\operatorname{supp}(a) = \{i \mid a_i \neq 0\}.$$

LEMMA 1. *Let $A = \{x_1, \dots, x_r\} + \mathbb{N}^p$ and $B = \{y_1, \dots, y_s\} + \mathbb{N}^p$ be two ideals of $\mathbb{N}^p$. Then*

$$A \cap B = \{x_i \vee y_j \mid i \in \{1, \dots, r\},\ j \in \{1, \dots, s\}\} + \mathbb{N}^p.$$

Proof. Set $C = \{x_i \vee y_j \mid i \in \{1, \dots, r\},\ j \in \{1, \dots, s\}\} + \mathbb{N}^p$.

If $x \in A \cap B$, then $x \geq x_i$ and $x \geq y_j$ for some $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$. Hence $x \geq x_i \vee y_j$ and thus $x \in C$.

Now take $x \in C$. Then $x \geq x_i \vee y_j$ for some i, j, whence $x \geq x_i$ and $x \geq y_j$, which leads to $x \in A \cap B$. ∎

PROPOSITION 2. *Let J be an ideal of $\mathbb{N}^p$. The following conditions are equivalent*:

(1) *J is irreducible.*

(2) *There exist $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, p\}$ and $k_1, \ldots, k_r \in \mathbb{N}$ such that*

$$J = \{k_1 e_{i_1}, \ldots, k_r e_{i_r}\} + \mathbb{N}^p.$$

Proof. (1) implies (2). Assume that $J = \{x_1, \ldots, x_r\} + \mathbb{N}^p$ and that $\#\mathrm{supp}(x_1) \geq 2$. Take $i, j \in \mathrm{supp}(x_1)$ with $i \neq j$. Then using Lemma 1 we get that $J = A \cap B$, where

$$A = \{x_1 - e_i, x_2, \ldots, x_r\} + \mathbb{N}^p, \qquad B = \{x_1 - e_j, x_2, \ldots, x_r\} + \mathbb{N}^p.$$

(2) implies (1). If $J = A \cap B$ with A, B properly containing J, then there exist x and y in the set of minimal elements of A and B, respectively, such that $x, y \notin J$. Note that $x \vee y \in (A \cap B) \setminus J$ (the coordinates of x and y corresponding to $i_1, \ldots, i_r$ must be less than $k_1, \ldots, k_r$, respectively, and thus $x \vee y$ fulfills the same condition). ∎

Remark 3. Observe that, in view of Proposition 2, if I and J are irreducible ideals of $\mathbb{N}^p$, then so is $I \cup J$ (note that $I \cup J$ is the ideal generated by the union of the generating systems for I and J; the fact that $I \cup J$ is an ideal is one of the main differences between ideal theory in semigroups and in rings).

The following consequence is not only a result of existence of a decomposition of an ideal into irreducibles, its proof gives the procedure for computing such decomposition.

COROLLARY 4. *Let J be an ideal of $\mathbb{N}^p$. Then there exist $J_1, \ldots, J_s$ irreducible ideals of $\mathbb{N}^p$ such that $J = J_1 \cap \cdots \cap J_s$.*

Proof. Assume that $J = \{x_1, \ldots, x_r\} + \mathbb{N}^p$. If J is not irreducible, then by Proposition 2, there exist j, k, l such that $k, l \in \mathrm{supp}(x_j)$ and $k \neq l$. If $x_j = \sum_{i=1}^{p} \lambda_i e_i$., then by Lemma 1, $J = J_1 \cap J_2$, with $J_1 = \{x_1, \ldots, x_j - \lambda_k e_k, \ldots, x_r\} + \mathbb{N}^p$ and $J_2 = \{x_1, \ldots, x_j - \lambda_l e_l, \ldots, x_r\} + \mathbb{N}^p$. We repeat the process for J_1 and J_2, and after a finite number of steps we get $J = J_{i_1} \cap \cdots \cap J_{i_l}$ with J_{i_k} fulfilling that its minimal elements have support with only one element, which by Proposition 2 means that they are irreducible ideals of $\mathbb{N}^p$. ∎

EXAMPLE 5. Let I be the ideal of $\mathbb{N}^2$ generated by $\{(3,0), (1,2), (0,5)\}$. Then

$$\begin{aligned} I &= \left(\{(3,0),(1,0),(0,5)\} + \mathbb{N}^2\right) \cap \left(\{(3,0),(0,2),(0,5)\} + \mathbb{N}^2\right) \\ &= \left(\{(1,0),(0,5)\} + \mathbb{N}^2\right) \cap \left(\{(3,0),(0,2)\} + \mathbb{N}^2\right). \end{aligned}$$

The rest of the section is devoted to provide us with an alternate characterization of irreducible ideals of $\mathbb{N}^p$. This new characterization will be generalized in the next sections for arbitrary monoids.

For an element x in a monoid S, define

$$\mathrm{B}(x) = \{y \in S \mid y \leq x\}.$$

Observe that in this setting $S \setminus B(x)$ is an ideal of S.

PROPOSITION 6. *Let $J = \{k_1 e_{i_1}, \dots, k_r e_{i_r}\} + \mathbb{N}^p$ be an ideal of $\mathbb{N}^p$. Assume that $\{j_1, \dots, j_s\} = \{1, \dots, p\} \setminus \{i_1, \dots, i_r\}$ and set for all $n \in \mathbb{N}$*

$$t_n = (k_1 - 1)e_{i_1} + \cdots + (k_r - 1)e_{i_r} + n(e_{j_1} + \cdots + e_{j_s}).$$

Then

$$J = \bigcap_{n \in \mathbb{N}} (\mathbb{N}^p \setminus \mathrm{B}(t_n)).$$

Proof. Take $s \in J$. Then for all $n \in \mathbb{N}$ we have that $s \nleq t_n$ and thus $s \notin \mathrm{B}(t_n)$, whence $s \in \bigcap_{n \in \mathbb{N}} (\mathbb{N}^p \setminus \mathrm{B}(t_n))$.

Now assume that $s = (x_1, \dots, x_p) \notin J$. Then $x_{i_1} < k_1, \dots, x_{i_r} < k_r$. Taking $n \in \mathbb{N}$ so that $n > \max\{x_{j_1}, \dots, x_{j_s}\}$, we obtain that $s \in \mathrm{B}(t_n)$ and therefore $s \notin \bigcap_{n \in \mathbb{N}} (S \setminus \mathrm{B}(t_n))$. ∎

Observe that, for any x, y in a monoid S, if $x \leq y$, then $\mathrm{B}(x) \subseteq \mathrm{B}(y)$. This implies that the sequence $\{\mathbb{N}^p \setminus \mathrm{B}(t_n)\}_{n \in \mathbb{N}}$ in Proposition 6 is a decreasing sequence (strictly decreasing in the case $r < p$) of ideals. Using this and Proposition 2 we obtain the following consequence.

COROLLARY 7. *Let J be an irreducible ideal of $\mathbb{N}^p$. Then there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements in $\mathbb{N}^p$ such that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ and so that*

$$J = \bigcap_{n \in \mathbb{N}} (\mathbb{N}^p \setminus \mathrm{B}(x_n)).$$

3. SOME IRREDUCIBLE IDEALS

We show that the ideals appearing in Proposition 6 and Corollary 7 are always irreducible ideals, even in a more general setting.

PROPOSITION 8. *Let S be a monoid and take $a \in S$. Then $S \setminus \mathrm{B}(a)$ is an irreducible ideal of S.*

Proof. We already know that $S \setminus \mathrm{B}(a)$ is an ideal of S. For proving that $S \setminus \mathrm{B}(a)$ is irreducible, just take into account that every ideal properly containing $S \setminus \mathrm{B}(a)$ must contain a. ∎

The reader acquainted with ideal ring theory will find the following result (and its proof) particularly odd.

LEMMA 9. *Let I be an ideal of a monoid S. The following conditions are equivalent.*

(1) *I is irreducible.*

(2) *For all $a, b \in S$, if $(a + S) \cap (b + S) \subseteq I$, then either $a + S \subseteq I$ or $b + S \subseteq I$.*

Proof. (1) implies (2). Assume that $(a + S) \cap (b + S) \subseteq I$, $a + S \not\subseteq I$, and $b + S \not\subseteq I$. Then $A = I \cup (a + S)$ and $B = I \cup (b + S)$ are two ideals that strictly contain I. In addition, $A \cap B = I \cup ((a + S) \cap (b + S)) = I$, which implies that I is not irreducible.

(2) implies (1). Assume that I is not irreducible. Then $I = A \cap B$ for A, B ideals of S properly containing I. Take $a \in A \setminus I$ and $b \in B \setminus I$. Then $(a + S) \cap (b + S) \subseteq A \cap B = I$, but neither $a + S \subseteq I$ nor $b + S \subseteq I$. ∎

PROPOSITION 10. *Let S be a monoid and let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of irreducible ideals of S such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} I_n$ is an irreducible ideal of S.*

Proof. In view of Lemma 9, it suffices to prove that if $(a + S) \cap (b + S) \subseteq \bigcap_{n \in \mathbb{N}} I_n$, then either $a + S \subseteq \bigcap_{n \in \mathbb{N}} I_n$ or $b + S \subseteq \bigcap_{n \in \mathbb{N}} I_n$.

Since $(a + S) \cap (b + S) \subseteq \bigcap_{n \in \mathbb{N}} I_n$, then $(a + S) \cap (b + S) \subseteq I_n$ for all $n \in \mathbb{N}$. Thus by Lemma 9, either $a + S \subseteq I_n$ or $b + S \subseteq I_n$. If $a + S \subseteq I_n$ for all n, then we are done. Assume that $a + S \not\subseteq I_k$ for some $k \in \mathbb{N}$ (and that k is the least nonnegative integer fulfilling this condition); then since $\{I_n\}_{n \in \mathbb{N}}$ is a decreasing sequence, we obtain that $a + S \not\subseteq I_{k+l}$ for all $l \in \mathbb{N}$. Using Lemma 9, we get $b + S \subseteq I_{k+l}$ for all $l \in \mathbb{N}$ and thus $b + S \subseteq I_j$ for all $j \in \{0, \ldots, k - 1\}$. Therefore $b + S \subseteq \bigcap_{n \in \mathbb{N}} I_n$. ∎

4. IRREDUCIBLE IDEALS AND DECOMPOSITION INTO IRREDUCIBLES

In this section we prove that, in the finitely generated case, the irreducible ideals are like the ones appearing in Proposition 10. First we give a generalization of the decomposition of irreducibles already studied on $\mathbb{N}^p$ (Corollaries 4 and 7; see the preliminaries to recall the definition of $\mathrm{E}(I)$).

THEOREM 11. *Let I be an ideal of the monoid $S = \langle s_1, \ldots, s_p \rangle$. Assume that $\mathrm{E}(I) = \bigcap_{i=1}^{r} J_i$, with J_i an irreducible ideal of $\mathbb{N}^p$ for all i (see Corollary*

4), *and that* $J_i = \bigcap_{n\in\mathbb{N}}(S\setminus \mathrm{B}(x_n^i))$ *for some ascending sequence* $\{x_n^i\}_{n\in\mathbb{N}} \subseteq \mathbb{N}^p$ (*see Corollary* 7). *Then*

$$I = \bigcap_{i=1}^{r}\left(\bigcap_{n\in\mathbb{N}}\left(S\setminus \mathrm{B}\left(\varphi\left(x_n^i\right)\right)\right)\right).$$

Proof. Set $K = \bigcap_{i=1}^{r}(\bigcap_{n\in\mathbb{N}}(S\setminus \mathrm{B}(\varphi(x_n^i))))$. In view of Propositions 8 and 10, K is an intersection of the irreducible ideals $\bigcap_{n\in\mathbb{N}}(S\setminus \mathrm{B}(\varphi(x_n^i)))$.

Take $s\in I$. Assume that there exist i,k such that $s\notin S\setminus \mathrm{B}(\varphi(x_k^i))$. Then $s\in \mathrm{B}(\varphi(x_k^i))$, or in other words, there exists $t\in S$ such that $s+t=\varphi(x_k^i)$. Since φ is surjective, $s=\varphi(a)$ and $t=\varphi(b)$ for some $a,b\in\mathbb{N}^p$. Hence $\varphi(a+b)=\varphi(x_k^i)$. Clearly $x_k^i\notin \mathrm{E}(I)$, whence $a+b\notin \mathrm{E}(I)$, which leads to $a\notin \mathrm{E}(I)$, contradicting the fact that $\varphi(a)=s\in I$. Hence $s\in S\setminus \mathrm{B}(\varphi(x_k^i))$ for all i,k and consequently $I\subseteq K$.

Now assume that $s\notin I$. Take $a\in\mathbb{N}^p$ such that $\varphi(a)=s$. This implies that $a\notin \mathrm{E}(I)$ and thus there exist i,k such that $a\notin S\setminus \mathrm{B}(x_k^i)$, that is, $a\in \mathrm{B}(x_k^i)$. Hence $x_k^i = a+b$ for some $b\in\mathbb{N}^p$ and therefore $\varphi(a)+\varphi(b)=\varphi(x_k^i)$, which leads to $s\in \mathrm{B}(\varphi(x_k^i))$, whence $s\notin K$. We have proved that $K=I$. ∎

Observe that by Proposition 8 and 10, the expression of I in Theorem 11 is just a decomposition of I into irreducibles of S. As a consequence of this result we obtain the following description of the irreducible ideals of a finitely generated monoid S.

COROLLARY 12. *Let S be a finitely generated monoid. The following conditions are equivalent.*

(1) *I is an irreducible ideal of S.*

(2) *There exists a sequence $\{s_n\}_{n\in\mathbb{N}}\subseteq S$ such that $I=\bigcap_{n\in\mathbb{N}}(S\setminus B(s_n))$ and $s_n\le s_{n+1}$ for all $n\in\mathbb{N}$.*

Proof. (1) implies (2). By Theorem 11, I can be expressed in the form

$$I = \bigcap_{i=1}^{r}\left(\bigcap_{n\in\mathbb{N}}\left(S\setminus \mathrm{B}\left(\varphi\left(x_n^i\right)\right)\right)\right).$$

Since I is irreducible, there exists $i\in\{1,\dots,r\}$ such that $I=\bigcap_{n\in\mathbb{N}}(S\setminus \mathrm{B}(\varphi(x_n^i)))$. In addition the elements $x_n^i\le x_{n+1}^i$ and thus $\varphi(x_n^i)\le\varphi(x_{n+1}^i)$. Taking $s_n=\varphi(x_n^i)$ we obtain (2).

(2) implies (1). The ideals $S\setminus \mathrm{B}(s_n)$ are irreducible by Proposition 8. Since $s_n\le s_{n+1}$ we get that $\mathrm{B}(s_n)\subseteq \mathrm{B}(s_{n+1})$, whence $S\setminus \mathrm{B}(s_{n+1})\subseteq S\setminus \mathrm{B}(s_n)$. Proposition 10 states that $\bigcap_{n\in\mathbb{N}}(S\setminus \mathrm{B}(s_n))$ is an irreducible ideal of S. ∎

Given an ideal I of a finitely generated monoid S, to compute a decomposition of I into irreducibles, one has to find first $\mathrm{E}(I)$. Note that the decomposition of I in S is tightly related to the decomposition of $\mathrm{E}(I)$ in $\mathbb{N}^p$. In particular, if $\mathrm{E}(I)$ is irreducible, then so is I. However, as the following example shows, it may happen that I is an irreducible ideal of S but $\mathrm{E}(I)$ is not irreducible.

EXAMPLE 13. Let $S = \mathbb{N}^2/\sigma$, where σ is the congruence on $\mathbb{N}^2$ generated by $\{((2,0),(0,3))\}$ (in this setting φ is the natural projection: $\varphi(x) = [x]$). Take $I = \{[(1,2)]\} + S$. Then $\mathrm{E}(I) = \{(3,0),(1,2),(0,5)\} + \mathbb{N}^2$ (in the next section we show how to compute $\mathrm{E}(I)$), which is the ideal of Example 5. Thus

$$\mathrm{E}(I) = \left(\{(1,0),(0,5)\} + \mathbb{N}^2\right) \cap \left(\{(3,0),(0,2)\} + \mathbb{N}^2\right)$$

and, by Proposition 6, we obtain that

$$\mathrm{E}(I) = \left(\mathbb{N}^2 \setminus \mathrm{B}((0,4))\right) \cap \left(\mathbb{N}^2 \setminus \mathrm{B}((2,1))\right).$$

Theorem 11 asserts that

$$I = \left(S \setminus \mathrm{B}([(0,4)])\right) \cap \left(S \setminus \mathrm{B}([(2,1)])\right)$$

(here $x_n^1 = (0,4)$ and $x_n^2 = (2,1)$ for all $n \in \mathbb{N}$) and since $(0,4)\sigma(2,1)$ we have that $I = S \setminus \mathrm{B}([(0,4)])$. Hence, by Proposition 8, I is an irreducible ideal of S.

5. SOME COMPUTATIONAL ASPECTS

In this section we describe a method for computing the decomposition into irreducibles of an ideal I of a finitely generated monoid S (once we know a presentation of the monoid and a system of generators of the ideal). The procedure consists in making effective the expression obtained in Theorem 11. To do that, we first show how to compute $\mathrm{E}(I)$, and then we focus on the computation of the ideals $\bigcap_{n \in \mathbb{N}} (S \setminus \mathrm{B}(\varphi(x_n^i))) = S \setminus (\bigcup_{n \in \mathbb{N}} \mathrm{B}(\varphi(x_n^i)))$.

Assume that S is generated by $\{s_1, \dots, s_r\}$ and that $\varphi\colon \mathbb{N}^p \to S$ is defined as above. Take ρ to be a canonical system of generators of σ (and in particular it is a presentation of S). In the rest of the section we identify S with $\mathbb{N}^p/\sigma$, and thus $\varphi(x)$ with $[x]_\sigma$. Assume that the ideal I is generated by $\{[\lambda_1], \dots, [\lambda_r]\}$. Then

$$\rho_I = \left\{(\lambda_1, \lambda_2), \dots, (\lambda_1, \lambda_r), (\lambda_1 + e_1, \lambda_1), (\lambda_1 + e_p, \lambda_1)\right\}$$

is a presentation for $S/\mathscr{R}_I$. Observe that in general $\mathrm{E}(I)$ is larger than the ideal of $\mathbb{N}^p$ generated by $\{\lambda_1, \ldots, \lambda_r\}$. Take κ to be a canonical system of generators of σ_I (recall that $S/\mathscr{R}_I$ is isomorphic to $\mathbb{N}^p/\sigma_I$) with respect to a given linear admissible order $\preceq$.

The following two results are the key for the computation of $\mathrm{E}(I)$. Since $\mathrm{E}(I)$ is an ideal of $\mathbb{N}^p$, it suffices to know the set $\mathrm{Minimals}_{\leq} \mathrm{E}(I)$.

LEMMA 14. *Let S, I, ρ, σ, ρ_I, σ_I, and κ be as above. If $x \in \mathrm{E}(I)$, $x - \beta \in \mathbb{N}^p$, and $x - \beta + \alpha \in \mathrm{Minimals}_{\leq} \mathrm{E}(I)$ for some $(\alpha, \beta) \in \kappa$, then $x = \mu \vee \beta$, for some $\mu \in \mathrm{Minimals}_{\leq} \mathrm{E}(I)$.*

Proof. Since $x \in \mathrm{E}(I)$, there exists $\mu \in \mathrm{Minimals}_{\leq} \mathrm{E}(I)$ such that $x - \mu \in \mathbb{N}^p$. By hypothesis, $x - \beta$ is also in $\mathbb{N}^p$ and thus $x = (\mu \vee \beta) + z$ for some $z \in \mathbb{N}^p$. As $\mu \in \mathrm{E}(I)$ and $\mathrm{E}(I)$ is an ideal of $\mathbb{N}^p$, we get that $\mu \vee \beta \in \mathrm{E}(I)$. The fact that $(\alpha, \beta) \in \sigma_I$ leads to $((\mu \vee \beta) - \beta + \alpha, \mu \vee \beta) \in \sigma_I$, and this implies that $[(\mu \vee \beta) - \beta + \alpha]_\sigma \in I$, whence $(\mu \vee \beta) - \beta + \alpha \in \mathrm{E}(I)$. Finally $x - \beta + \alpha = ((\mu \vee \beta) - \beta + \alpha) + z \in \mathrm{Minimals}_{\leq} \mathrm{E}(I)$, which implies that $z = 0$, because $(\mu \vee \beta) - \beta + \alpha$ already belongs to $\mathrm{E}(I)$. ∎

PROPOSITION 15. *Let S, I, ρ, σ, ρ_I, σ_I, κ, and $\preceq$ be as above. If $\mathrm{Minimals}_{\leq} \mathrm{E}(I) = \{\mu_1 \prec \cdots \prec \mu_s\}$, then $\mu_{k+1} = (\mu_i \vee \beta) - \beta + \alpha$, for some $(\alpha, \beta) \in \kappa$ and $i \in \{1, \ldots, k\}$.*

Proof. Since $\mu_{k+1} \in \mathrm{E}(I)$, then $\mathrm{NF}_\kappa(\mu_{k+1}) = \mathrm{NF}_\kappa(\mu_1) = \mu_1$. Hence there exists $(\alpha, \beta) \in \kappa$ such that $\mu_{k+1} - \alpha \in \mathbb{N}^p$. Set $\mu_{k+1} - \alpha + \beta = x \in \mathbb{N}^p$. Since $(\alpha, \beta) \in \sigma_I$ and $\mu_{k+1} - \alpha \in \mathbb{N}^p$, we get $[x]_{\sigma_I} = [\mu_{k+1} - \alpha + \beta]_{\sigma_I} = [\mu_{k+1}]_{\sigma_I}$, which leads to $[x]_\sigma \in I$, or in other words, $x \in \mathrm{E}(I)$. Applying Lemma 14 to x we obtain that $x = (\mu_i \vee \beta)$ for some $i \in \{1, \ldots, s\}$. In addition

$$\mu_i \preceq \mu_i \vee \beta \preceq (\mu_i \vee \beta) - \beta + \alpha = \mu_{k+1},$$

which implies $i < k + 1$. ∎

With these two results we give the following algorithm for computing $\mathrm{E}(I)$.

Algorithm 16. Let S, I, ρ, σ, ρ_I, σ_I, κ, and $\preceq$ be as above.

Input. The set $\{[\lambda_1], \ldots, [\lambda_r]\}$ (the system of generators of I) and a canonical system of generators $\kappa = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\}$ of the congruence σ_I on $\mathbb{N}^p$ with respect to a given linear admissible order $\preceq$.

Output. The set $\mathrm{Minimals}_{\leq} \mathrm{E}(I)$.

(1) Compute $\mu_1 = \mathrm{NF}_\kappa(\lambda_1)$.

(2) Set $A = \{\mu_1\}$.

(3) Compute $B = \{(a \vee \beta_j) - \beta_j + \alpha_j \mid a \in A,\ j \in \{1,\dots,t\}\}$.

(4) Compute $C = B \cap \mathrm{Minimals}_{\leq}\{x \in \mathbb{N}^p \mid \mathrm{NF}_\kappa(x) = \mu_1\}$ (note that an element $b \in B$ is in $\mathrm{Minimals}_{\leq}\{x \in \mathbb{N}^p \mid \mathrm{NF}_\kappa(x) = \mu_1\}$ if and only if, for all $y < b$, $\mathrm{NF}_\kappa(y) \neq \mu_1$).

(5) If $C \subseteq A$, then return A.

(6) $A := A \cup C$; go to Step 3.

Once we know how to compute $\mathrm{E}(I)$, following the steps imposed by Theorem 11, we must find out how to compute the ideals of the form $S \setminus (\bigcup_{n \in \mathbb{N}} \mathrm{B}(\varphi(x_n))) = \mathbb{N}^p/\sigma \setminus (\bigcup_{n \in \mathbb{N}} \mathrm{B}([x_n]))$. First of all let us find the sets $\mathrm{B}([x])$.

LEMMA 17. *Let $x \in \mathbb{N}^p$. Then*

$$\mathrm{B}([x]) = \{[a] \mid \text{exists } b \in [x] \text{ such that } a \leq b\}.$$

Proof. Assume that $[a] \in \mathrm{B}([x])$. Then $[a] \leq [x]$, whence $[a] + [c] = [x]$ for some $c \in \mathbb{N}^p$. Hence $a + c \in [x]$ and clearly $a \leq a + c$.

If $a \leq b$ and $b \in [x]$, then $[a] \leq [b] = [x]$. ∎

Next we introduce some definitions and auxiliary results needed to simplify notation.

For a given $n \in \mathbb{N}$, let π_i and Π_i, $1 \leq i \leq n$, be the maps

$$\pi_i\colon (\mathbb{N} \cup \{\infty\})^n \to (\mathbb{N} \cup \{\infty\}), \qquad \pi_i(x_1,\dots,x_n) = x_i,$$

$$\Pi_i\colon (\mathbb{N} \cup \{\infty\})^n \to (\mathbb{N} \cup \{\infty\})^{n-1},$$

$$\Pi_i(x_1,\dots,x_n) = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n).$$

For a given congruence σ on $\mathbb{N}^p$, let $\Pi_i(\sigma)$ be the congruence on $\mathbb{N}^{p-1}$ defined by

$$(a_1,\dots,a_{i-1},a_{i+1},\dots,a_p)\Pi_i(\sigma)(b_1,\dots,b_{i-1},b_{i+1},\dots,b_p)$$

if there exists $d, d' \in \mathbb{N}$ such that

$$(a_1,\dots,a_{i-1},d,a_{i+1},\dots,a_p)\sigma(b_1,\dots,b_{i-1},d',b_{i+1},\dots,b_p).$$

It can be shown that if $\{(\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t)\}$ is a system of generators of σ, then $\{(\Pi_i(\alpha_1),\Pi_i(\beta_1)),\dots,(\Pi_i(\alpha_t),\Pi_i(\beta_t))\}$ is a system of generators of $\Pi_i(\sigma)$ (see [9] or [8]).

Given an ascending sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}^p$, we can take into account each of the p sequences $\{\pi_i(x_n)\}_{n \in \mathbb{N}}$, $1 \leq i \leq p$, which are again ascending sequences in $\mathbb{N}$ and thus they have a limit $z_i \in \mathbb{N} \cup \{\infty\}$. The limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$ is just the element of $(\mathbb{N} \cup \{\infty\})^p$ whose coordinates are $(z_1,\dots,z_p)$.

LEMMA 18. *Assume that* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{x'_n\}_{n\in\mathbb{N}}$ *have the same limit. Then*

$$\bigcup_{n\in\mathbb{N}} \mathrm{B}([x_n]) = \bigcup_{n\in\mathbb{N}} \mathrm{B}([x'_n]).$$

Proof. Assume that $[a] \in \bigcup_{n\in\mathbb{N}} \mathrm{B}([x_n])$. Then there exists $k \in \mathbb{N}$ such that $[a] \in \mathrm{B}([x_k])$. Hence $[a] + [b] = [x_k]$ for some $b \in \mathbb{N}^p$. Since both $\{x_n\}_{n\in\mathbb{N}}$ and $\{x'_n\}_{n\in\mathbb{N}}$ have the same limit, there exists $m \in \mathbb{N}$ such that $x_k \le x'_m$ (at some point the coordinates of $\{x'_n\}_{n\in\mathbb{N}}$ must be greater than or equal to the coordinates of $\{x_n\}_{n\in\mathbb{N}}$). Thus $[a] \le [x_k] \le [x'_n]$, which leads to $[a] \in \cup_{n\in\mathbb{N}} \mathrm{B}([x'_n])$. The other inclusion is proved in the same way. ∎

If we have a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}^p$ and $z \in (\mathbb{N} \cup \{\infty\})^p$ is its limit, then in the following we will write $\mathrm{B}([z])$ instead of $\bigcup_{n\in\mathbb{N}} \mathrm{B}([x_n])$. Using this notation, by Theorem 11, associated with an ideal I of $\mathbb{N}^p/\sigma$ there exist $z^1, \ldots, z^r \in (\mathbb{N} \cup \{\infty\})^p$ such that

$$I = \bigcap_{i=1}^{r} \left(\mathbb{N}^p/\sigma \setminus \mathrm{B}([z^i])\right).$$

Observe that the sequences $\{x_n^i\}_{n\in\mathbb{N}}$ appearing in Theorem 11 are not arbitrary, they all come from the decomposition into irreducibles of $\mathrm{E}(I)$ as obtained in Corollary 4 and Proposition 6. Thus once we have computed $\mathrm{E}(I)$, the elements z^i can be easily calculated.

Hence, the next step for the computation of the decomposition into irreducibles passes through the calculation of $\mathrm{B}([z])$, with z the limit of an ascending sequence on $\mathbb{N}^p$. We may find several difficulties while computing this set. The first one is then even if $z \in \mathbb{N}^p$ the set $[z]$ need not be finite in general, and thus Lemma 17 is not easily applied. The other one is what to do in the case some of the coordinates of z are infinite. These two problems are the ones we are going to study next, and as we will see they are somehow connected.

In view of Lemma 17 it seems reasonable to find out how to compute the σ-class of an element $a \in \mathbb{N}^p$. Recall that we are assuming that ρ is a canonical system of generators of σ with respect to a given linear admissible order $\preceq$ on $\mathbb{N}^p$. Suppose that $\rho = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\}$. We already know how to check whether an element $b \in \mathbb{N}^p$ belongs to $[a]$: one only has to compute $\mathrm{NF}_\rho(a)$ and $\mathrm{NF}_\rho(b)$ and see whether we obtain the same result. By the definition of NF_ρ, this is equivalent to saying that there exists a sequence $i_1, \ldots, i_q$ of elements in $\{1, \ldots, t\}$ (they do not need to be all different) such that

(1) $b + \sum_{k=1}^{q}(-\alpha_{i_k} + \beta_{i_k}) = \mathrm{NF}_\rho(a)$,

(2) $b + \sum_{k=1}^{r}(-\alpha_{i_k} + \beta_{i_k}) - \alpha_{i_{r+1}} \in \mathbb{N}^p$ for all $r \in \{0, \ldots, q-1\}$.

This is the same as to say that b "rewrites" to $\mathrm{NF}_\rho(a)$ (using the rewriting rule given by ρ). If we look backward in this rewriting process, we obtain the following result.

LEMMA 19. *The element $b \in \mathbb{N}^p$ belongs to $[a]$ if and only if there exists a sequence $(\alpha_{i_1}, \beta_{i_1}), \dots, (\alpha_{i_q}, \beta_{i_q})$ of elements of ρ such that*

(1) $\mathrm{NF}_\rho(a) + \sum_{k=1}^{q}(-\beta_{i_k} + \alpha_{i_k}) = b$,
(2) $\mathrm{NF}_\rho(a) + \sum_{k=1}^{r}(-\beta_{i_k} + \alpha_{i_k}) - \beta_{i_{r+1}} \in \mathbb{N}^p$ *for all* $r \in \{0, \dots, q-1\}$.

This result provides us with a method for obtaining recurrently the elements of $[a]$. This procedure is implemented in the following algorithm.

Algorithm 20. Let S, I, ρ, σ, ρ_I, σ_I, κ, and $\preceq$ be as above.
Input. ρ and $a \in \mathbb{N}^p$.
Output. $[a]$ in case it has finitely many elements; otherwise the algorithm never stops and constructs recurrently on A the set $[a]$.

(1) Compute $\mathrm{NF}_\rho(a)$.
(2) Set $A := B := \{\mathrm{NF}_\rho(a)\}$.
(3) While $B \neq \emptyset$
choose $u \in B$,
set $B := (B \setminus \{u\}) \cup \{u - \beta_j + \alpha_j \mid u - \beta_j \in \mathbb{N}^p, j \in \{1, \dots, t\}\}$,
set $A := A \cup \{u - \beta_j + \alpha_j \mid u - \beta_j \in \mathbb{N}^p, j \in \{1, \dots, t\}\}$.
(4) Return A.

The following result gives us the key to deciding whether the class $[a]$ is or is not finite. Note that, in the case where $[a]$ is finite, $[a] = \mathrm{Minimals}_{\leq} [a] = \mathrm{Maximals}_{\leq} [a]$.

LEMMA 21. *The set $[a]$ has infinitely many elements if and only if in some execution of the while loop of Algorithm 20 we obtain that $\mathrm{NF}_\rho(a) + x \in A$ for some $x \in \mathbb{N}^p \setminus \{0\}$.*

Proof. *Necessity.* If $[a]$ has infinitely many elements, then, applying Dickson's lemma, there exist $b \in [a]$ and $x \in \mathbb{N}^p \setminus \{0\}$ such that $b + x \in [a]$. Hence $\mathrm{NF}_\rho(a) \sigma b$ and $(\mathrm{NF}_\rho(a) + x) \sigma (b + x)$, which leads to $\mathrm{NF}_\rho(a) \sigma \mathrm{NF}_\rho(a) + x$. Since Algorithm 20 computes the set $[a]$ recurrently, at some stage, it must happen that $\mathrm{NF}_\rho(a) + x \in A$.

Sufficiency. Clearly $\mathrm{NF}_\rho(a) + kx \in [a]$ for all $k \in \mathbb{N}$ and thus $[a]$ has infinitely many elements. ∎

With this result we can modify Algorithm 20 to obtain a new algorithm that returns $[a]$ if this set has finitely many elements; otherwise it says that $[a]$ has infinitely many elements. It suffices to add a new line in the while loop as follows:

> Check whether for any $x \in A$, the element $x - \mathrm{NF}_\rho(a) \in \mathbb{N}^p \setminus \{0\}$. If so, return "The set $[a]$ has infinitely many elements."

LEMMA 22. *Assume that $\{x_n\}_{n \in \mathbb{N}}$ is an ascending sequence of elements in $\mathbb{N}^p$ whose limit is $z = (z_1, \ldots, z_p) \in \mathbb{N}^p$, and that $z + x \in [z]$ with $x \neq 0$. Without loss of generality assume that* $\mathrm{supp}(x) = \{1, \ldots, r\}$. *Set $y_n = x_n + n(e_1 + \cdots + e_r)$ for all $n \in \mathbb{N}$. Then*

$$\bigcup_{n \in \mathbb{N}} \mathrm{B}([x_n]) = \bigcup_{n \in \mathbb{N}} \mathrm{B}([y_n]),$$

or in other words, if $z' = (\infty, \ldots, \infty, z_{r+1}, \ldots, z_p)$, then $\mathrm{B}([z]) = \mathrm{B}([z'])$.

Proof. First note that, since $z + x \in [z]$, we have that $[z + nx] = [z]$ for all $n \in \mathbb{N}$.

From the definition of y_n, we get that $\mathrm{B}([x_n]) \subseteq \mathrm{B}[(y_n)]$, whence $\bigcup_{n \in \mathbb{N}} \mathrm{B}([x_n]) \subseteq \bigcup_{n \in \mathbb{N}} \mathrm{B}([y_n])$.

Now take $[a] \in \mathrm{B}([y_n])$; then $[a] \leq [y_n] = [x_n + n(e_1 + \cdots + e_r)]$. In addition $x_n \leq z$ and this leads to $x_n + n(e_1 + \cdots + e_r) \leq z + n(e_1 + \cdots + e_r)$. Taking σ-classes we obtain $[x_n + n(e_1 + \cdots + e_r)] \leq [z + n(e_1 + \cdots + e_r)]$. In addition, $z + n(e_1 + \cdots + e_r) \leq z + nx$. We conclude that $[a] \leq [x_n + n(e_1 + \cdots + e_r)] \leq [z + n(e_1 + \cdots + e_r)] \leq [z + nx]$ and thus $[a] \leq [z]$, which leads to $[a] \in \mathrm{B}([z]) = \bigcup_{n \in \mathbb{N}} \mathrm{B}([x_n])$.

This result is telling us that in the case where $z \in \mathbb{N}^p$ and the σ-class of z is infinite, we translate the problem to a new problem with limit not belonging to $\mathbb{N}^p$. Next we show how to get rid of the coordinates which are equal to infinity in the limit of the sequence $\{x_n\}_{n \in \mathbb{N}}$.

LEMMA 23. *Assume that $\{x_n\}_{n \in \mathbb{N}}$ is an ascending sequence of elements in $\mathbb{N}^p$ whose limit is $z = (z_1, \ldots, z_p) \in (\mathbb{N} \cup \{\infty\})^p$ and $z_1 = \infty$. Then $[a]_\sigma \in \mathrm{B}([z]_\sigma)$ if and only if $[\Pi_1(a)]_{\Pi_1(\sigma)} \in \mathrm{B}([\Pi_1(z)]_{\Pi_1(\sigma)})$.*

Proof. Assume that $[a]_\sigma \in \bigcup_{n \in \mathbb{N}} \mathrm{B}([x_n]_\sigma)$. Then there exist $k \in \mathbb{N}$ and $c \in \mathbb{N}^p$ such that $[a + c]_\sigma = [x_k]_\sigma$. Hence $[\Pi_1(a) + \Pi_1(c)]_{\Pi_1(\sigma)} = [\Pi_1(x_k)]_{\Pi_1(\sigma)}$.

Assume now that $[\Pi_1(a)]_{\Pi_1(\sigma)} \in \bigcup_{n \in \mathbb{N}} \mathrm{B}([\Pi_1(x_n)]_{\Pi_1(\sigma)}$.) This implies that there exists $k \in \mathbb{N}$ and $(c_2, \ldots, c_p) \in \mathbb{N}^{p-1}$ such that $\Pi_1(a) + (c_2, \ldots, c_p) \Pi_1(\sigma) \Pi_1(x_k)$. Take $d, d' \in \mathbb{N}$ such that $de_1 + \Pi_1(a) + (0, c_2, \ldots, c_p) \sigma d' e_1 + \Pi_1(x_k)$. Note that we can choose $d \geq \pi_1(a)$ and thus there exists $c_1 \in \mathbb{N}$ satisfying $\pi_1(a) + c_1 = d$. Since $z_1 = \infty$, we can find $m \in \mathbb{N}$ fulfilling that $\pi_1(x_m) \geq d'$. Finally we have that $a + (c_1, \ldots, c_p) \sigma d' e_1 + \Pi(x_k) \leq x_m$. ∎

Using this last result for $p = 1$, we obtain that if $z_1 = \infty$, then $\mathrm{B}([z])$ is the whole monoid $\mathbb{N}/\sigma$. With all this, we are ready to give an algorithm to compute the set $\mathrm{Maximals}_{\leq}(\mathrm{E}(\mathrm{B}[z]))$ which fully describes $\mathrm{B}([z])$ for a given z limit of an ascending sequence on $\mathbb{N}^p$.

Algorithm 24. Let S, I, ρ, σ, ρ_I, σ_I, κ, and $\preceq$ be as above.
Input. $z \in (\mathbb{N} \cup \{\infty\})^p$, the limit of an ascending sequence on $\mathbb{N}^p$.
Output. $\mathrm{Maximals}_{\leq}(\mathrm{E}(\mathrm{B}([z])))$.

(1) If $z \in \mathbb{N}^p$, apply Algorithm 20 to z. If $[z]$ is finite, then return $[z]$. Otherwise proceed as in Lemma 22; that is, if $z + x \in [z]$ (the element x is determined in Algorithm 20), then make all the coordinates of z that are in $\mathrm{supp}(x)$ equal to ∞.

(2) Take the least $i \in \{1, \ldots, p\}$ such that $z_i = \infty$. Compute $\Pi_i(z)$ (and $\Pi(\sigma)$, recalculate a canonical system of generators for $\Pi(\sigma)$ needed for Algorithm 20) and apply this algorithm with entry $\Pi_i(z)$. If A is the output of the algorithm then return

$$\left\{(a_1, \ldots, a_{i-1}, \infty, a_{i+1}, \ldots, a_p) \mid (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_p) \in A\right\}.$$

Once we know how to compute $\mathrm{B}([z]) = \bigcup_{n \in \mathbb{N}} \mathrm{B}([x_n])$ we are interested in computing which are the expressions of its associated ideal, that is, $\mathrm{E}(\mathbb{N}^p/\sigma \setminus \mathrm{B}([z]))$. Observe that $\mathrm{E}(\mathbb{N}^p/\sigma \setminus \mathrm{B}([z])) = \mathbb{N}^p \setminus \mathrm{E}(\mathrm{B}([z]))$. We give now a method to describe this set. Since $\mathrm{E}(\mathbb{N}^p/\sigma \setminus \mathrm{B}([z]))$ is an ideal of $\mathbb{N}^p$, it suffices to give a system of generators for it. Assume that $\{m_1, \ldots, m_r\} \subset (\mathbb{N} \cup \{\infty\})^p$ are the maximals of the set $\mathrm{E}(\mathrm{B}([z]))$. An element $(a_1, \ldots, a_p)$ belongs to $\mathbb{N}^p \setminus \mathrm{E}(\mathrm{B}([z]))$ if and only if it belongs to $\bigcap_{i=1}^r \mathbb{N}^p \setminus \{x \in \mathbb{N}^p \mid x \nleq m_i\}$. Lemma 1 shows how to perform intersections of ideals of $\mathbb{N}^p$. Thus we only have to explain how to compute for a given $k = (k_1, \ldots, k_p) \in (\mathbb{N} \cup \{\infty\})^p$ the set of minimal elements of $\mathbb{N}^p \setminus \{x \in \mathbb{N}^p \mid x \nleq k\}$. If $x \nleq k$, then there exists $i \in \{1, \ldots, p\}$ such that $x_i > k_i$ and therefore $k_i \neq \infty$. Hence we get that

$$\mathbb{N}^p \setminus \{x \in \mathbb{N}^p \mid x \nleq k\} = \bigcup_{\substack{i=1 \\ k_i \neq \infty}}^{p} \{(0, \ldots, 0, k_i + 1, 0, \ldots, 0)\} + \mathbb{N}^p.$$

This is the last step needed to give a procedure for computing the decomposition into irreducibles of an ideal of a finitely generated cancellative monoid (assuming that we are given a system of generators of the ideal). Summarizing, the steps to follow are:

(1) Compute

$$I = \bigcap_{i=1}^{r} \left(\bigcap_{n \in \mathbb{N}} \left(\mathbb{N}^p/\sigma \setminus \mathrm{B}\left(\left[x_n^j\right]\right)\right)\right) = \bigcap_{i=1}^{r} \left(\mathbb{N}^p/\sigma \setminus \mathrm{B}[z^i]\right).$$

(2) For all $i \in \{1, \ldots, r\}$ compute

$$\text{Maximals}\big(\text{E}\big(\text{B}([z^i])\big)\big).$$

(3) For all $i \in \{1, \ldots, r\}$ compute

$$\text{E}\big(\mathbb{N}^p/\sigma \setminus \text{B}([z^i])\big),$$

which tells you which one is $\mathbb{N}^p/\sigma \setminus \bigcup_{n \in \mathbb{N}} \text{B}([x_n^i])$.

EXAMPLE 25. Let σ be the congruence generated by

$$\{((5,0,0,0),(0,7,0,0)),((0,0,6,0),(0,0,1,0))\},$$

let S be the monoid $\mathbb{N}^4/\sigma$, and let I be the ideal $[(3,3,6,5)] + S$. First, using Algorithm 16, we compute $\text{Minimals}_{\leq} \text{E}(I)$, obtaining that this set is equal to $\{(8,0,1,5),(0,10,1,5),(3,3,1,5)\}$ and therefore

$$\text{E}(I) = \{(8,0,1,5),(0,10,1,5),(3,3,1,5)\} + \mathbb{N}^4.$$

By Corollary 4, we have that the decomposition into irreducibles of $\text{E}(I)$ is

$$\begin{aligned} \text{E}(I) = {} & (\{(0,0,0,5)\} + \mathbb{N}^4) \cap (\{(0,0,1,0)\} + \mathbb{N}^4) \\ & \cap (\{(0,10,0,0),(3,0,0,0)\} + \mathbb{N}^4) \\ & \cap (\{(8,0,0,0),(0,3,0,0)\} + \mathbb{N}^4). \end{aligned}$$

Now, using Proposition 6, we get that the decomposition of $\text{E}(I)$ into irreducibles is equal to

$$\begin{aligned} & \Big(\mathbb{N}^4 \setminus \bigcup_{n \in \mathbb{N}} \text{B}(x_n^1)\Big) \cap \Big(\mathbb{N}^4 \setminus \bigcup_{n \in \mathbb{N}} \text{B}(x_n^2)\Big) \\ & \cap \Big(\mathbb{N}^4 \setminus \bigcup_{n \in \mathbb{N}} \text{B}(x_n^3)\Big) \cap \Big(\mathbb{N}^4 \setminus \bigcup_{n \in \mathbb{N}} \text{B}(x_n^4)\Big), \end{aligned}$$

with $x_n^1 = (n,n,n,4)$, $x_n^2 = (n,n,0,n)$, $x_n^3 = (2,9,n,n)$, and $x_n^4 = (7,2,n,n)$. Thus, by Theorem 11, we get

$$\begin{aligned} I = {} & (\mathbb{N}^4/\sigma \setminus \text{B}[z^1]) \cap (\mathbb{N}^4/\sigma \setminus \text{B}[z^2]) \\ & \cap (\mathbb{N}^4/\sigma \setminus \text{B}[z^3]) \cap (\mathbb{N}^4/\sigma \setminus \text{B}[z^4]) \end{aligned}$$

with $z^1 = (\infty,\infty,\infty,4)$, $z^2 = (\infty,\infty,0,\infty)$, $z^3 = (2,9,\infty,\infty)$, and $z^4 = (7,2,\infty,\infty)$.

We now compute $\text{Maximals}(\text{E}(\text{B}([z^i])))$ for $i = 1,2,3,4$.

Since $z^1 \notin \mathbb{N}^4$ and its first, second, and third coordinates are equal to ∞. Algorithm 20 says that we have to compute $\Pi_1(\Pi_1(\Pi_1(\sigma)))$, which is the trivial congruence on $\mathbb{N}$. Further, $\Pi_1(\Pi_1(\Pi_1(z^1))) = 4$, whence

$$\mathrm{Maximals}\big(\mathrm{E}\big(\mathrm{B}([z^1])\big)\big) = \{(\infty, \infty, \infty, 4)\}.$$

The calculation of Maximals(E(B($[z^2]$))) is very similar to the previous case. We obtain

$$\mathrm{Maximals}\big(\mathrm{E}\big(\mathrm{B}([z^2])\big)\big) = \{(\infty, \infty, 0, \infty)\}.$$

For $i = 3$, we compute $\Pi_3(\Pi_3(\sigma))$, which is equal to the congruence τ generated by

$$\{((5,0),(0,7))\}.$$

We also have that $\Pi_3(\Pi_3(z^3)) = (2,9)$, an element whose τ-class is equal to $\{(2,9),(7,2)\}$. Thus

$$\mathrm{Maximals}\big(\mathrm{E}\big(\mathrm{B}([z^3])\big)\big) = \{(7,2,\infty,\infty),(2,9,\infty,\infty)\}.$$

The case $i = 4$ is very similar to the case $i = 3$. For this case the result is the same as for $i = 3$,

$$\mathrm{Maximals}\big(\mathrm{E}\big(\mathrm{B}([z^4])\big)\big) = \{(7,2,\infty,\infty),(2,9,\infty,\infty)\}.$$

Consequently, we deduce that I is not irreducible and that

$$I = \big(\mathbb{N}^4/\sigma \setminus \mathrm{B}([z^1])\big) \cap \big(\mathbb{N}^4/\sigma \setminus \mathrm{B}([z^2])\big) \cap \big(\mathbb{N}^4/\sigma \setminus \mathrm{B}([z^3])\big).$$

Using the remark after Algorithm 24 we get that

$$S \setminus \mathrm{B}([z^1]) = [(0,0,0,5)] + S,$$
$$S \setminus \mathrm{B}([z^2]) = [(0,0,1,0)] + S,$$

and

$$S \setminus \mathrm{B}([z^3]) = \{[(8,0,0,0)],[(3,3,0,0)],[(0,10,0,0)]\} + S,$$

the decomposition of I can also be expressed as

$$I = ([(0,0,0,5)] + S) \cap ([(0,0,1,0)] + S)$$
$$\cap(\{[(8,0,0,0)],[(3,3,0,0)],[(0,10,0,0)]\} + S).$$

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